On the relativity of magnitudes
Delboeuf's forgotten contribution to the 19th century problem of space

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ARTICLE INFO

ABSTRACT

Faced with the mathematical possibility of non-Euclidean geometries, 19th Century geometers were tasked with the problem of determining which among the possible geometries corresponds to that of our space. In this context, the contribution of the Belgian philosopher-mathematician, Joseph Delboeuf, has been unduly neglected. The aim of this essay is to situate Delboeuf's ideas within the context of the philosophies of geometry of his contemporaries, such as Helmholtz, Russell and Poincaré. We elucidate the central thesis, according to which Euclidean geometry is given special status on the basis of the relativity of magnitudes, we uncover its hidden history and show that it is defensible within the context of the philosophies of geometry of the epoch. Through this discussion, we also develop various ideas that have some relevance to present-day methods in gravitational physics and cosmology.

Dieser Stoff kann also vorgestellt werden als ein physischer Raum; dessen Punkte sich in dem geometrischen bewegen.

[Bernhard Riemann *Gravitation und Lichts, 1853*]

Introduction

Following the discovery of the mathematical possibility of non-Euclidean geometries by Lobachevsky (1829a, 1829b), the "problem of space", that is, the problem of determining which among the available geometries should be chosen as that which represents the space of our physical world, drew the attention of pre-relativistic physicists, philosophers and mathematicians throughout the 19th Century. The well known contributions of Helmholtz (1870, 1876), Riemann (1854), Poincaré (1898) and Lie (1893) brought together an assemblage of empiricist and neo-Kantian ideas which would give birth to new perspectives, such as geometrical conventionalism. On one hand, neo-Kantian strands of thought suggested that the geometry of space should be something regarded as distinct from the material contents therein; on the other hand, empiricists argued that geometry is only an abstraction from the observed behaviours of material bodies. By considering space as a *condition for the possibility of measurement* rather than a condition for the possibility of experience, Helmholtz developed a form of *empiricist neo-Kantianism*, which had a profound and enduring influence on later neo-Kantians and logical positivists (Friedman, 2001, 2009; Ryckman, 2003).

There are, however, some problems with this standard 19th Century approach to the problem of space: Helmholtz's "conditions for the possibility of measurement" rely on an approximately physically instantiated notion of rigid bodies, which was undermined by later developments in special and general relativity. Whereas Poincaré's lesser emphasis on empiricism arguably avoids this issue, the Helmholtzian views which he adopts appear to be inconsistent with certain other aspects of his philosophy of space (see Section 3.3).

The aim of this paper is to introduce English-speaking audiences to Joseph Delboeuf's alternative approach to the space problem, which has been overlooked by recent accounts such as Biagioli et al. (2016), Dewar and Eisenthal (2020), Heinzmann (2001). Although Delboeuf's work engages deeply with the Kantian tradition, he is not even mentioned in recent neo-Kantian texts in the philosophy of physics such as Biagioli et al. (2016), Bitbol, Kerszberg, and Petitot (2009), Friedman (2001, 2014/1983), Ryckman (2005). The most significant acknowledgement of Delboeuf's contribution since the mid 20th Century is by Torretti (2012/1978). But while Toretti recognises that Delboeuf was "probably the earliest philosophical writer who had first-hand acquaintance with the works of Lobachevsky" (p.153), and acknowledges Delboeuf's ideas as "interesting" (p.298), he ultimately gives them a disfavourable verdict, and even revives a fallacious objection to Delboeuf's philosophy of geometry which had formerly been touted by Russell and Poincaré (see Section 3.2).

Delboeuf defended the apriority of Euclidean space on the basis of the relativity of magnitudes. Although he developed his ideas independently, the central argument dates back all the way to Wallis

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(1696) (see Section 2.2.1), and has recently been revived by Čulina (2018, 2020, 2023). Delboeuf is unique in that he gives this notion of the relativity of magnitude the status of "first postulate", thereby attempting to erect geometry upon new foundations. Delboeuf rejected the prior systems of geometry of Euclid, Lobachevsky, Mill and Kant on the basis that none of these had rationally deduced the postulates or axioms taken as primary. If geometry is the study of determinations in space, Delboeuf claims that our concept of space is characterised by the notion of "homogeneity" by which he means not only that the properties of space are the same in all its parts, but also that these are independent of its size (Delboeuf, 1860, p.126). In other words, in geometry we must be able to consider the shape and size of figures independently. For Delboeuf, this notion of homogeneity is not an empirical fact about some "real space" (Delboeuf, 1893), rather, it is the result of the process of abstraction which the intellect undertakes in its effort to describe nature in terms of universal laws, i.e. to step outside of the contingency of given (real) material objects.

While Delboeuf's philosophy of science is a fascinating topic in its own right, we will not dive too deeply into this topic here. Our discussion will be mostly limited to situating this author's ideas concerning geometry within their historical context in the 19th Century and drawing parallels with ideas of other philosophers and mathematicians. Among the novel arguments and findings made in this essay: (1) We uncover a convergence of thought between Delboeuf's approach and an essay by Leibniz titled *Uniformis locus* which has only been made available relatively recently by De Risi (2005), De Risi (2007) (see Section 2.3). (2) we elucidate the fundamental difference between Helmholtz's empirically realisable motions, and Delboeuf's symmetry conditions, and argue that only the latter are relevant to the Kantian conception of space (see Section 3.1). (3) We refute Bertrand Russell's enduring '*relative angles*' objection to Delboeuf's notion of the relativity of magnitudes (see Section 3.2). (4) We propose that Poincaré's "principle of relative motion" and his more general conception of the mathematical infinite can be grounded in Delboeuf's philosophy (see Section 3.3). We conclude in Section 4 with some suggestions concerning the relevance of these findings to the methods of modern physics. Since our findings here demonstrate the incompatibility of the non-Euclidean geometries with the relativity of magnitude, this work may be seen as complementary to the recent work of Eisenthal (2024) which brings attention to the incompatibility of non-Euclidean geometries of constant curvature with the relativity of velocity.

Given that Poincaré's philosophy of space has some significant similarities to Delboeuf's, is better-known than Delboeuf's, and represents in some sense a lucid synthesis of many of the findings that were made by investigators in the late 19th Century, we have chosen in Section 1 to set the scene by discussing certain key aspects of Poincaré's philosophy of space that do not directly involve the question of geometry, but which are nonetheless deeply relevant to later discussions.

1. Preliminary remarks on Poincaré on space

While Poincaré is quite famous today for his geometrical conventionalism (which will be discussed in Section 2.1), other aspects of his philosophy of space and mathematics are equally significant and will help provide context for our subsequent discussion of geometry. In this section, we will briefly cover some of these aspects: (1) Poincaré's revision of the Kantian notion of synthetic a priori reasoning, (2) Poincaré's empiricist account of the distinction between changes of state and changes of place, (3) the distinction between empirical objects and their mathematical idealisations.

1.1. Mathematical reasoning

By the late 19th Century, Kant's claim that mathematics contains synthetic a priori propositions was being subjected to severe criticism by the logicists. Frege and Russell attempted to show that all true

mathematical statements could be derived from a basic set of concepts defined terminologically. Thus all mathematical truths would be reduced to logic, without need of intuitions.[1] Poincaré, on the other hand, did not abandon the notion of synthetic a priori reasoning, but rather, attempted to revise it.[2]

Poincaré develops his conception of synthetic a priori knowledge in the first chapter of *Science and Hypothesis*, titled *On the Nature of Mathematical Reasoning*. Kant had claimed that the basic propositions of arithmetic, such as $5 + 7 = 12$, are *synthetic*, since nowhere in the concept of the sum of 5 and 7 is contained the concept of 12. Something additional is needed for Kant, that is, an intuition of space in which the two quantities can be placed side-by-side with one another, and the operation of summation can be accomplished (Kant, 2004/1783, p.18). Such a claim is controversial, in part because it depends greatly upon how we define things. It is not too difficult to define our numbers in such a way that basic propositions of arithmetic, such as $5 + 7 = 12$ or $2+2 = 4$ appear as analytic truths. Unlike Kant, Poincaré does not argue that there is anything synthetic in these basic propositions; instead, he claims that the so-called "demonstrations" of these sums are really only analytic "verifications". However, these trivial verifications are not the true subject matter of mathematics, on the contrary, Poincaré (2015/1913, p.33) insists:

> It may even be said the very object of the exact sciences is to spare us these direct verifications.

The essence of mathematics, for Poincaré, lies in the ability to generalise across an infinity of cases, using what is called "reasoning by recurrence", or "mathematical induction". The basic structure of a proof by induction proceeds as follows:

1. The theorem is proven for $n = 1$.
2. It is shown that if it is true for $n = a$, it must be true for $n = a+1$.
3. Therefore we know that it is true for $n = 2$, and likewise $n = 3, 4, 5\dots$. By induction, we have shown that it must be true for all $n \in \mathbb{Z}$.

This enables one to make generalisations about some theorem over an infinity of cases.[3] It is in this possibility of reasoning by recurrence—which Poincaré calls "the mathematical reasoning *par excellence*"—that he locates the true *synthetic a priori* judgment (Poincaré, 2015/1913, p.39):

> This rule, inaccessible to analytic demonstration and to experience, is the veritable type of the synthetic a priori judgment. [...] Mathematical induction, that is, demonstration by recurrence, [...] imposes itself necessarily because it is only the affirmation of a property of the mind itself.

1.2. Empirical ground of space

In chapter IV of *Science and Hypothesis*, titled *Space and Geometry*, Poincaré takes up the perspective of a naive investigator attempting to make sense of the world present to his senses while lacking any pre-conceived notions about how these ought to be organised and interpreted. How do we come to the idea of space, and in particular, how do we distinguish between changes of position and changes of state (such as changes in colour)? Poincaré presents this problem as follows (Poincaré, 2015/1913, p.70):

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[1] Note that for Frege, this meant that mathematics would be purely analytic, whereas Russell viewed logic as synthetic.

[2] See Folina (2016/1992) for an in depth discussion of Poincaré's neo-Kantianism.

[3] In Section 3.3, we will argue that it is not the notion of infinity, but the *a priori* concept of symmetry or mathematical equality that makes reasoning by recurrence possible.

Whether an object changes its state or merely its position, this is always translated for us in the same manner: by a modification in an aggregate of impressions. How then could we have been led to distinguish between the two?

His solution is rather straight forward:

It is easy to account for. If there has only been a change of position, we can restore the primitive aggregate of impressions by making movements which replace us opposite the mobile object in the same relative situation. We thus correct the modification that happened and we reestablish the initial state by an inverse modification.

A change in spatial position is distinguished from a change in state by the possibility of performing the reverse operation by means of the correlative movement of our own bodies. In the case of sight, this movement may also be performed by the "appropriate movement of the eyeball."

Now the possibility that certain "aggregates of impressions" may be restored through our correlative movements depends upon the existence of "solid bodies", i.e. bodies which retain the relations among their parts while changing position with respect to us. Indeed, it is the observation of solid bodies, Poincaré argues, that has taught us to distinguish between changes of state and changes of position, such that, he concludes:

if there were no solid bodies in nature, there would be no geometry.

1.3. Mathematical idealisation

While this empiricist account of the origin of geometry is persuasive, it does not lead directly to the complete *mathematical* notion of geometric space. While I may observe that as a body recedes from me, I can restore its original size by approaching it once more, I can never infer from experience that this will continue to be true if the body recedes to an arbitrarily large distance. Should my concept of space therefore be limited to distances for which the compensation is practically realisable? Clearly, when we imagine objects in geometrical space, we do not limit ourselves to distances which our bodies are capable of traversing. Rather, we consider space as potentially infinite in extent, and thereby we implicitly imagine an *idealised* observer capable of visiting all parts of this space at will to perform the necessary compensatory motions. The space of geometry differs in this respect from the empirical condition that motivated its invention.

It is even more clear that an *idealisation* is involved when we place our representations in space–time.[4] In space–time we imagine bodies, extended in time as well as in space, and thereby we implicitly invoke the possibility of an idealised observer that may travel to the different parts of this space–time and measure it with ideal rulers and clocks. But these motions are by no means physically realisable. It is not even possible to visit distant points which seem to lie in our plane of simultaneity, let alone to travel into the past. If geometry can, in any sense, be said to have an empirical origin, it must have departed from its empirical roots in order to encompass the notion of time.

A second issue concerning the difference between the empirical and mathematical concepts of space is that no empirically given "natural solid" is absolutely rigid. When we look at any given body close enough we find motion and change in its structure, modifications due to heat, the vibrations of the constituent particles, and so on. These contingencies make it impossible for us to use empirical objects as standards for the definition of a mathematical space. As Poincaré (2015/1913, p.79) puts it:

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[4] The present comments concerning space–time are not drawn directly from Poincaré's works, however, they are inspired by his discussions of the issues.

Geometry would be only the study of the movements of solids; but in reality it is not occupied with natural solids, it has for object certain ideal solids, absolutely rigid, which are only a simplified and very remote image of natural solids.

To reach the mathematical concept of space, we must substitute our empirical notions of solid bodies with their *ideal* counterparts; and in so doing we substitute the empirically grounded, physical concept of space – which we have no definite knowledge of – with the pure, mathematical concept of space, of which we have absolute knowledge *a priori*. Once we have performed this substitution of the empirical objects with their ideal counterparts we make it possible to apply the mathematical reasoning which Poincaré characterises as *synthetic a priori*: While we may attempt to infer by *physical* induction that a rock remains the same wherever it is placed in relation to the other bodies of the universe, this knowledge will only ever be approximate, contingent and subject to the possibility of being refuted by experience. However, if my object is not a rock, but an ideal rigid body in Euclidean space, I can say with absolute certainty that it will retain the relations amongst its parts no matter where it is placed in this space.

2. The problem of space's geometry

In the first part of this section, we briefly recount the well-known history of Helmholtz and Riemann's canonical 19th Century approach to the problem of spatial geometry which was based on the notion of the free mobility of bodies 2.1. In the latter parts (2.2 and 2.3), we discuss the alternative approach of Delboeuf and some others which is based on the idea of the relativity of magnitudes.

2.1. The axiom of free mobility

The repeated failures to prove the necessity of Euclid's fifth postulate on the basis of the first four culminated in Lobachevsky's construction of a self-consistent geometry based on the denial of the parallel postulate (Lobachevsky, 1829a). Just as theorems concerning shapes in Euclidean geometry can be studied and proven, a corresponding set of theorems pertaining the Lobachevsky's hyperbolic geometry can be proven mathematically. Which set of theorems is, then, true of our space? This glaring ambiguity at the level of mathematics prompted various thinkers, including Riemann and Helmholtz, to seek an empirical ground for the validity of Euclidean geometry, or lack thereof.

2.1.1. Riemann
Following from Gauss' work on the geometry of curved surfaces, Riemann developed the general concept of a "multiply extended manifold" whose curvature may vary from point to point. Since the metrical properties of this manifold should be grounded in empirical facts, this manifold needed to be susceptible of measurement, which implied the mobility of certain quantities in space (Riemann, 1854):

Measuring involves the superposition of the quantities to be compared; it therefore requires a means of transporting one quantity to be used as a standard for the others.

The first hypothesis that Riemann explores is that "the length of lines is independent of their configuration, so that every line can be measured by every other". This allows for a broad class of possible geometries that we now know as *Riemannian geometries*. Riemann also remarked that if we assume – not only that lines are independent of configuration – but also that the bodies are so, then:

it follows that the curvature is everywhere constant, and the angle sum in all triangles is determined if it is known in one.

2.1.2. Helmholtz

Helmholtz placed great emphasis on this latter idea, arguing that the mobility of rigid bodies was a necessity for the possibility of measurement, and concluded that only the spaces of constant curvature could properly be considered as geometry (Helmholtz, 1866, 1870). Helmholtz acknowledges, however, that the natural bodies apparent to observation are never identical to our idealisations of these. In his latter paper (Helmholtz, 1870), Helmholtz approaches something like a Kantian view, according to which the notion of a geometric figure would be "formed independently of actual experience". However, Helmholtz insists that:

> we should have to maintain that the axioms of geometry are not synthetic propositions, as Kant held them: they would merely define what qualities and deportment a body must have to be recognised as rigid.

Rather than being a *condition for the possibility of representation*, like Kant thought, space and its geometry become *conditions for the possibility of measurement*, as it were.[5] Helmholtz finishes his essay by settling on a conventionalist stance according to which: if taken apart from mechanical propositions, the axioms of geometry "constitute a form into which any empirical content whatever will fit". However, this is not only true of Euclid's axioms, but also of the axioms of spherical and pseudospherical geometry (Helmholtz, 1870).[6]

2.1.3. Poincaré

Following Helmholtz, Poincaré further developed this conventionalist standpoint (Poincaré, 1898, 1905/1902). According to Poincaré any empirical assertion of some given geometry over another is founded on a "disguised definition"; It is a *convention*, and only an appeal to some extra-empirical theory virtue such as "*simplicity*" may allow us to decide between conventions. Poincaré's preference for conventionalism, rather than a purer form of Kantianism was grounded in his *group-theoretic* approach to the problem. For Poincaré space is not a form of the sensibility, since "sensations by themselves have no spatial character", rather the "sensible space" must be a *form of our understanding*: "it is an instrument which serves us not to represent things to ourselves, but to reason upon things" (Poincaré, 1898).

However, the geometry of this form cannot be determined a priori for Poincaré, since there are a multiplicity of conceivable forms that we may use to reason on things. These are the *group structures*, which are the objects of study of mathematics. The various transformations on a Euclidean space are only one among a multitude of possible group structures that may be employed if experience warrants it. From this point of view, nothing truly distinguishes Euclidean geometry from the alternatives apart from its simplicity. The empirical observation that it is at least approximately instantiated in the observable behaviours of natural solids and rays of light does not even distinguish it from geometries of very low curvature.

Despite claiming the conventionality of geometry, Poincaré devotes very little attention in his work to geometries of changing curvature. The bulk of Poincaré's discussion of the conventionality of geometry in part II on *Space* (chapters III, IV and V) of *Science and Hypothesis* concerns the geometries of constant curvature. The reason for his neglect of the former is given in the one passage in which they are briefly discussed (Poincaré, 2015/1913, p.63):

> most of these definitions are incompatible with the motion of a rigid figure, [...] These geometries of Riemann, in many ways so interesting, could never therefore be other than purely analytic and would not lend themselves to demonstrations analogous to those of Euclid.

Like Helmholtz, Poincaré rejects Riemann's geometries of changing curvature on the basis that they are incompatible with the motion of rigid figures. However, for Poincaré the crucial point here is that this incompatibility would undermine the very aim of mathematics. Since the geometrical properties of figures in a space of changing curvature would depend upon the value of the curvature from place to place, it would become hopeless to make those *inductive generalisations* that Poincaré views as so central to mathematical reasoning. Particular propositions about these geometries would not be synthetic, but *analytic*, since they would depend upon how the curvature is defined to change from point to point.

2.2. The relativity of magnitudes I: before Delboeuf

Helmholtz and Poincaré's refutations of Kant's Euclidean a priori rest essentially on a single claim: that the geometries of constant positive or negative curvature of Riemann and Lobachevsky respectively may just as well serve as *forms* into which the empirical content of our sensations may be placed. These geometries, they say, have just the same right to be viewed as "transcendentally given" as that of Euclid (Helmholtz, 1870).

But is there not some characteristic of Euclidean space, beyond its mere "simplicity", that sets it apart from those of constant non-zero curvature? Indeed there is. It is that *Euclidean space remains similar to itself at different scales*. In other words, we may *zoom into* some part of this space without changing anything about it. Thinking in terms of figures, rather than space itself, Euclidean space is the only space which allows for the possibility of incongruent *similar* figures (i.e. figures which differ in size but possess the same shape). All other geometries necessarily fail this test since curvature is a scale-dependent property of space. For instance, the sum of the angles of a triangle placed in Lobachevsky's hyperbolic space will shrink as the triangle is enlarged with respect to this space; therefore, two equilateral triangles of different sizes will not be similar.[7]

This criterion, by which Euclidean space can be uniquely determined, seems to have only been considered a handful of times in the history of geometry. For us, it is easy to become conscious of it, since we have knowledge of non-Euclidean geometry, and we can thereby easily identify what characteristic distinguishes Euclidean geometry by contrast. Prior to the development of the theory of non-Euclidean geometries however, it would have been more difficult to deduce the relationship between the absence of an absolute scale and the parallel postulate.

2.2.1. Wallis (1663)

The relationship between the possibility of similarities and the parallel postulate was first recognised by the English mathematician John Wallis, Savilian Chair at Oxford, in 1663 (although his proof was published in 1696 (Wallis, 1696)), over a century prior to the development of the theory of non-Euclidean geometries. Wallis attempted to show that Euclid's fifth postulate can be deduced from ideas which are self-evident. Though his proof is usually regarded as yet another failed historical attempt to prove the parallel postulate, we will see that his argument is quite significant and profound.[8]

Wallis's proof is in two parts:[9]

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[5] This view is elaborated by Russell (1898).

[6] The "pseudospherical" geometry is Helmholtz's term for the hyperbolic geometry of Lobachevsky and Bolyai.

[7] Throughout this paper, we use the term "similar" in the geometrical sense to denote figures of different sizes which have the same shape; we also sometimes refer to shape-preserving scale-transformations as "transformations by similarity" or simply "similarities".

[8] See for instance Jammer (2013, p.145) for a characterisation of Wallis's proof as a failed attempt to prove the parallel postulate.

[9] The original text is written in Latin by Wallis. We will not go through the details of Wallis's demonstration here; the reader can consult this in Hill (1925) for a reconstruction of the proof in English. See also Therrien (2020) for a more detailed discussion of Wallis's proof.

1. Firstly, Wallis demonstrates that Euclid's fifth postulate is identical to the possibility of constructing *similar* triangles, that is, triangles which have the same shape though they differ in size.
2. Secondly, Wallis provides a metaphysical argument for the possibility of transformations by similarity. This is due to the distinction between quality and quantity. Whereas, for Wallis, the size of a figure is a *quantity*, the shape of a figure belongs to the category of *quality*. Being different categories, each of these two must be able to vary independently of the other.

2.2.2. Carnot and Laplace

The next mention of this relationship between the possibility of similar figures and the parallel postulate appears in a note in Carnot's *Géometrie de Position* (Carnot, 1803, p.481):[10]

> The theory of parallels depends on a more primary notion which appears to me to be of the same order of clarity as that of the perfect equality or superposition of figures; this is the notion of *similarity*. It seems to me that it can be regarded as a self-evident principle, that which exists as large, such as a ball, a house, a drawing, can be made in small, and vice-versa; by consequence, whatever figure we may imagine, it is possible to imagine others of all sizes and similar to the first, that-is-to-say of which all the dimensions have amongst themselves the same proportions as that of the first. This notion once admitted, it is easy to establish the theory of parallels, without recourse to the notion of infinity.

Though Carnot asserts that the proof is easy, he does not derive it. Moreover, he does not cite Wallis's proof, so it is not clear whether or not he learned of it from there.

This idea is also mentioned by Laplace in passing amid a discussion of the scale-invariance of the inverse-square law of gravitational attraction (Laplace, 1835, p. 471–472). Likewise in a footnote we find:[11]

> The perception of extension contains a special property, self-evident and without which we cannot rigorously establish the properties of parallels. The idea of a limited extension, for example of the circle, contains nothing which depends on its absolute size. But, if we diminish, by thought, its radius, we are inevitably inclined to diminish in the same ratio its circumference and the sides of all the figures inscribed. This proportionality appears to be a much more natural postulate than that of Euclid; it is curious to find it again in the results of universal gravity.

Once again, the work of Wallis is not mentioned.

2.3. The relativity of magnitudes II: Delboeuf

Joseph Delboeuf was a Belgian psychologist, mathematician, and philosopher. Although he spent the bulk of his career as an experimental psychologist, he obtained doctoral degrees in both philosophy and mathematics and was deeply concerned with the foundations of geometry in his youth. While he was studying philosophy at the University of Liège, his friend and colleague François Folie had attempted to prove the necessity of Euclid's parallel postulate. Folie's professor had pointed out the questionable proposition involved, and this disappointment led Folie to abandon the endeavour (Delboeuf, 1895, p.346). Delboeuf, on the other hand, did not abandon his youthful ambitions, and some years later published a radical reconception of geometry that would place Euclidean intuitions surely at its foundation (Delboeuf, 1860).[12]

The difficulty with Euclid's fifth postulate draws investigators into a labyrinth from which they can only escape by a total revolution in thinking about geometry. In this respect, two pathways are available; we may either (1) seek new foundations for Euclidean geometry, or (2) we should absorb Euclid's geometry into a more general conception, of which Euclid's is only a special case. The second approach, that of the "neo-geometers", has been favoured by history. Delboeuf, on the other hand, embarks upon the first project (Delboeuf, 1894b, p.122). Lobachevsky's discovery, for Delboeuf, did not disprove the necessity of the parallel postulate in geometry; rather, it only served to help us better understand what our Euclidean intuitions are founded on.

2.3.1. The homogeneity of space

In his *Prolégomènes Philosophiques De La Geometrie Et Solution Des Postulats* (henceforth *Prolégomènes*) of 1860, Delboeuf independently rediscovers the insight of Wallis, Carnot and Laplace that the "mutual independence of shape and size" implies the Euclidean nature of space (Delboeuf, 1860). He elevates this insight to the "first postulate" of geometry and argues, much like Wallis, that it is in philosophy and metaphysics, not geometry, that we must seek its justification. This justification is found, for Delboeuf, to be implicit in the concept we have of space, namely, that it is "homogeneous". Homogeneity, for Delboeuf, is a more restrictive criterion than what this word usually means today. Today, we generally recognise a manifold M as homogeneous if all points stand in the same relation to the whole. This can be specified mathematically in terms of the isometry of the metric g under the group action of the translation group T. Delboeuf calls this property "isogeneity" rather than homogeneity, and recognises that it holds for instance of the circumference of a circle, the surface of a sphere, as well as non-Euclidean geometries of constant curvature such as the hyperbolic geometry of Lobachevsky and the spherical geometry of Riemann. The word "homogeneity", for Delboeuf, is reserved for an extension which is *self-similar*, that is to say, invariant under scale transformations; in present day language we would call this *rescaling conformal isometry*: the group action ϕ transforms the metric g such that $\phi^*g = \Omega^2 g$, where Ω is constant across space. It can be shown that a geometry (M, g), which is isometric under translations as well as rescalings must be Euclidean (Wald, 2010).

For Delboeuf, the assertion of the homogeneity of space is not some habit of thought taught by the regularities of our experience, as an empiricist such as Mill might see it, on the contrary, Delboeuf sees the space and time of our experience as heterogeneous (Delboeuf, 1860, p.41)[13]:

> Any being does not remain the same when we transport it from here to there; today it is different from what it was yesterday and from what it will be tomorrow.

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[10] Translation of: "La théorie des parallèles tient à une notion première qui me paroît être à-peu-près du même ordre de clarté que celle de l'égalité parfaite ou de la superposition; c'est la notion de *similitude*. Il me semble qu'on peut regarder comme un principe de première évidence, que ce qui existe en grand, comme une boule, une maison, un dessin, peut être fait en petit et réciproquement; que part conséquent, quelque figure qu'on veille imaginer, il est possible d'en imaginer d'autres de toutes grandeurs et semblables à la première, c'est-à-dire dont toutes les dimensions aient entre elles les mêmes proportions que celles de la première. Cette notion une fois admise, il et facile détablire la théorie des parallèles, sans recourir à la notion de l'infini". (Emphasis in original).

[11] Translation of: "La perception de l'étendue renferme donc une propriété spéciale, évidente par elle-meme et sans laquelle on no peut rigoureusement établir les propriétés des parallèles. L'idée d'une étendue limitée, par exemple du cercle, ne contient rien qui dépende de sa grandeur absolue. Mais, si nous diminuons, par la pensée, son rayon, nous sommes portés invinciblement à diminuer dans le meme rapport sa circonférence et les côtes de toutes les figures inscrites. Cette proportionnalité me parait être un *postulatum* bien plus naturel que celui d'Euclide; il est curieux de la retrouver dans les résultats de la pesanteur universelle".

[12] François Folie would go on to become a distinguished astronomer and director of the Royal Observatory of Belgium in Uccle.

[13] Translation of: "Un être quelconque, ne reste pas le même quand d'ici on le transporte là ; aujourd'hui il est différent de ce qu'il était hier et de ce qu'il sera demain".

Instead, the notion of space's homogeneity follows from the *method* of science, it is the consequence of the effort of our intelligence to uncover the unity underlying the multiplicity given in phenomena[14]:

> It is from this vast whole that intelligence must seek the laws; it is this infinite variety of facts that she must reduce to a few general facts. For this, she resorts to a first abstraction; she supposes time and space to be perfectly homogeneous, that is to say, as we will explain later, indefinitely and arbitrarily divisible into parts which differ only by their magnitude.

Through this process of abstraction, of "ideal *homogeneification*" of space and time, we make the world appear as "inert" and "infinitely divisible" (p.42). But it is only insofar as we carry out this abstraction that we are able to subject this world to universal laws. In bk. I, ch. 2 of *Prolégomènes*, Delboeuf describes how the object of science changes across multiple consecutive processes of abstraction. Zoology and botany study organisms insofar as they are instances of their species, the physical and chemical sciences consider bodies as inert, they are no longer unique individuals but assemblages of some basic substances such as elements which are each universal. Through a further abstraction we reach the mathematical sciences: mechanics considers bodies in terms of the actions they exert on one-another, which we call *force*. Now if we abstract further, away from the changes and motions which result from the inequality of forces, Delboeuf writes (p.67)[15]:

> the universe is reduced to an ensemble of figures. The science of these figures is called geometry.

All figures are endowed with a shape and a size. Indeed geometry can be studied from either perspective. Delboeuf gives the name "*synthetic geometry*" to the study of the figure "in-itself", that is, if we "bring back questions of size to questions of shape". Conversely, he calls "*analytic geometry*" the study of shapes in terms of relations of magnitudes, for instance "the shape of a figure is given in terms of the length of the coordinates of each of its points" (p.69). The postulate of the mutual independence of shape and size is thus inherent to synthetic geometry, since it is assumed in the notion that one can study shape in itself. As well as the homogeneity of space, Delboeuf identifies this *first postulate* with the recognition of the relativity of magnitude (p.129)[16]:

> To say that space is homogeneous is at bottom nothing but the assertion that there is no absolute magnitude.

2.3.2. Leibniz and the definition of the straight

Interestingly, the concepts of homogeneity and isogeneity had previously been discussed by Leibniz under the names of "self-similarity" and "self-congruence" respectively. In an unpublished essay titled "*Uniformis locus*" which has so far been discussed solely in the work of Vincenzo de Rizi (De Risi, 2005; De Risi, 2007, 2015), Leibniz includes the definitions of the plane, the straight and space in terms of their self-similarity[17]:

A locus can be called uniform or self-congruent if its congruently bounded parts are congruent. On the other hand, a locus is self-similar if its similarly bounded parts are similar. The only self-similar loci are the straight line, the plane, and space itself. Uniform loci include all self-similar loci and, besides, others—that is to say, among the lines, the arc of a circle and the cylindrical helix and, among the surfaces, the spherical and the cylindrical ones.

Indeed Leibniz even recognises the same concept of "homogeneity" as Delboeuf, though he does not use the term "isogeneity", preferring to use "equality" instead:

> As I have discussed similarity and congruence, I have also distinguished between homogeneity and equality. In fact, the loci that can be transformed into similar ones are homogeneous; while the loci that can be transformed into congruent ones are equal.

The resemblance to Delboeuf's writings here is quite striking.[18] In fact, Delboeuf himself placed great emphasis on defining not only space, but the straight line and the plane in terms of their homogeneity or self-similarity (Delboeuf, 1860, p.180)[19]:

> *The plane is a homogeneous surface; the straight is a homogeneous line;* that is to say that a portion of a plane, *magnified*, generates the same plane; that a portion of a straight, *magnified*, reproduces the straight. We can therefore regard homogeneity as being the *genetic* characteristic of space, of the plane, and of the straight line.

As Delboeuf notes, historically there have been a multitude of apparently different definitions of the straight: viewed from the standpoint of *distance*, it is the shortest path between two points; from that of *direction*, it is a line of constant direction, and so on (p.175). Given one of these definitions, the others would appear as *synthetic* truths, but none can be used to deduce the others analytically. However, these *synthetic* theorems, Delboeuf argues, are each in fact *analytic* decompositions of the original intuition that gave rise to them all (p.177). If we wish to escape the paradoxes, to overcome the impossible task of deriving one definition from another, we must seek to characterise the fundamental *essence* of the straight or of the plane. This leads Delboeuf to define them in terms of their *homogeneity*; like Euclidean space, the straight and the plane are distinguished by their invariance under dilations. Although the original intuition which underlies our notion of the straight is not itself conceptual, it can be captured by a concept – that of *homogeneity* – from which the theory can be developed.

Given this definition, the notion of straight lines in non-Euclidean geometries automatically becomes absurd; the geodesics of a curved space will not be invariant under dilations since the space itself is not invariant under these transformations: a great circle on a sphere surface will be pushed outside of the sphere it inhabits if it is dilated with respect to that surface. The notion of geodesics which is proper to Riemannian geometry assumes one of the secondary definitions of the

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[14] Translation of: "C'est ce vaste ensemble dont l'intelligence doit chercher les lois c'est cette infinie variété de faits qu'elle doit ramener à quelques faits généraux. Pour cela, elle a recours à une première abstraction; elle suppose le temps et l'espace parfaitement homogènes, c'est-à-dire, comme nous l'expliquerons plus tard, indéfiniment et arbitrairement divisibles en parties qui ne diffèrent que par leur grandeur".

[15] Translation of : "l'univers se réduit à un ensemble de figures. La science des figures est la géométrie".

[16] Translation of: "Dire que l'espace est homogène, revient, au fond, à dire que rien n'a une grandeur absolue".

[17] The term "locus", which Leibniz uses, corresponds to the term "quantum" in Delboeuf's writings. We have also used the term "extension" in this text to denote the same concept.

[18] There is no evidence that Delboeuf was aware of the above mentioned essay by Leibniz, since it had not been published at the time. In fact, Delboeuf recounts that he was only alerted to the similarity between his work and some of Leibniz's other writings (which had just been made available two years prior in Leibniz (1858)) by his mentor Ueberweg shortly after the publication of Dlboeuf's book (Delboeuf, 1895, p.346). While it is plausible that Delboeuf may have been influenced indirectly by Leibniz through his conversations with Ueberweg, the similarity between Leibniz and Delboeuf should be understood first and foremost as an instance of convergence of thought, suggesting an affinity between these two thinkers.

[19] Translation of: "*Le plan est une surface homogène; La droite est une ligne homogène* ; c'est-à-dire qu'une portion de plan, *majorée*, engendre le même plan; qu'une portion de droite, *majorée*, reproduit la droite. Nous pouvons donc regarder l'homogénéité comme étant le caractère *génétique* de l'espace, du plan, de la droite".

straight: that it is *the shortest path between two points*, but if we follow Delboeuf and take the essence of the straight to be its *homogeneity*, we would no longer recognise the geodesics of non-euclidean spaces as straight lines, and the controversy concerning Euclid's postulate of parallels would be resolved. As De Risi (2015, p.61) puts it in his commentary on Leibniz's definition[20]:

> one could say that the truth of the Parallel Postulate is the condition for such a definition or rather that this last notion of a straight line, taken as a *real* definition, already implies the truth of the Parallel Postulate.

2.3.3. Neo-Kantianism?

While there are many aspects of Delboeuf's philosophy of geometry and science in general that could reasonably be characterised as neo-Kantian, Delboeuf explicitly made an effort to distinguish himself from Kant. The central argument of Delboeuf's book is framed in terms of a dialectic between Kant's apriorism and Mill's empiricism, which he seeks to reconcile by recognising the *process of abstraction* as the source of the apodicticity of the laws of nature (Delboeuf, 1860, p.50)[21]:

> It is therefore on an abstraction of our mind that the apodicticity of the laws of nature rests; and the famous axiom on the constancy and invariability of these same laws is itself only a consequence. [...] This solution reconciles empiricism and idealism, Mill and Kant.

Delboeuf's main objection to Kant's account of geometry is that it only "push[ed] back the difficulty instead of resolving it" since it makes the laws of geometry "the laws of our nature [...] without telling us why it is that precisely the ideas of geometry are innate to us" (p.8). As we have seen, Delboeuf seeks to answer this "why" by identifying homogeneity as the key concept that characterises this intuition, and from which the postulates of geometry would follow.

Like Kant, Delboeuf recognises the invariability and uniformity we discover in nature as the imprint of our own reasoning (Delboeuf, 1860, p.51)[22]:

> We are also getting closer to Kant; when we discover nature's agreement with the laws of our mind, we recover what we have placed in her: we have given her order and invariability; we discover order and invariability; we see what we are capable of seeing; we see nature as we are capable of seeing her. Abstraction is an operation of our mind, and nature has become abstract for us; it shows us straight lines, perfect circles; We discover simple and pure bodies, distilled water, potassium nitrate, carbonate of lime, whereas it provides us with mixtures, water from the sea and rivers, saltpeter, chalk and marble.

Concerning space, we have already seen that Delboeuf makes an effort to distinguish "real" or empirical space from "geometrical" space. In fact, in 1893, returning to the topic of geometry after a three decade hiatus, Delboeuf devotes an entire essay to emphasising this very distinction (Delboeuf, 1893). Concerning "real" space, Torretti (2012/1978) has brought attention to the following intriguing passage in which Delboeuf argues that the non-Euclidean geometries of constant curvature would in no ways help us to represent real space better than the Euclidean (Delboeuf, 1894a, p.372)[23]:

> We can therefore say of Riemann and Lobachevsky's spaces that they are artificial spaces, like Euclidean space; and in this respect they are just as geometrical as Euclidean space. But they have no special quality to represent real space better than the latter. This [real space] certainly has a curvature, but this curvature is different at each of its points and varies there at each instant. The real figures, that is to say, the bodies, change with time and place. The constant curvatures of meta-Euclidian spaces are therefore as far from reality as is the homogeneity of Euclidean space.

From the present-day perspective, we are compelled to respond: *why not then ditch Euclidean space, and the "meta-Euclidean" spaces of constant curvature? Why not embrace the varying curvature of real space and apply Riemann's broader notion of differentiable manifolds?* Delboeuf seems on the point of anticipating the revolutions of the subsequent decades, but instead he passes by this and retreats to his aprioristic defence of Euclidean geometry. Torretti (2012/1978, p.300) takes this as evidence that Delboeuf had not read Riemann. However, it is more likely that Delboeuf avoided considering Riemann's geometries of changing curvature for the very same reason that most commentators did at the time; that, as Poincaré put it, these geometries are purely *analytic*, they do not permit the kinds of mathematical generalisations that are the essence of all *a priori synthetic* reasoning.

Moreover, it is worth remarking that Riemann's own writings are not in contradiction with Delboeuf's position as Torretti seems to believe. Amid some of Riemann's most suggestive remarks, in which he proposes (in 1853) that the force of gravity be described along with inertia in terms of the dynamical geometry of a physical space, anticipating Einstein's equivalence hypothesis (Riemann, 1876), we find a similar distinction between a *physical* (or real) and a *geometrical* space that Delboeuf makes:[24]

> I seek the cause [of gravity] in the state of motion of the continuous substance spread throughout the entire infinite space. [...] this substance may be thought of as a physical space whose points move in geometrical space.

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[20] For more information concerning Leibniz's development of this notion of the self-similarity or homogeneity in geometry, see De Risi (2015).

[21] Translation of: "C'est donc sur une abstraction de notre esprit que repose l'apodicticité des lois de la nature ; et le fameux axiome sur la constance et l'invariabilité de ces mêmes lois, n'en est lui-même qu'une conséquence. [...] Cette solution concilie à la fois l'empirisme et l'idéalisme, Mill et Kant".

[22] Translation of: "Nous nous rapprochons aussi de Kant. En effet, quand nous constatons dans la nature l'accord avec les lois de notre esprit, nous *y* retrouvons ce que nous y avons mis nous *y* avons mis l'ordre et l'invariabilité ; nous *y* retrouvons l'ordre et l'invariabilité ; nous *y* voyons ce que nous pouvons *y* voir ; nous la voyons comme nous pouvons la voir. L'abstraction est une opération de notre esprit, et la nature est pour nous devenue abstraite elle nous montre des lignes droites, des cercles parfaits ; nous *y* découvrons des corps simples, des corps purs,de l'eau distillée,du nitrate de potasse, du carbonate de chaux, tandis qu'elle nous fournit des mélanges, l'eau de la mer et des fleuves, du salpêtre, de la craie et du marbre".

[23] Translation of: "On peut donc dire des espaces de Riemann et Lobatschewsky, que ce sont des espaces artificiels, comme l'espace euclidien ; et sous ce rapport, ils sont tout aussi géométriques que l'espace euclidien. Mais ils n'ont pas qualité spéciale pour représenter mieux que lui l'espace réel. Celui-ci, comme je l'ai dit dans ma première étude, a certainement une courbure, mais cette courbure est différente en chacun de ses points et *y* varie à chaque instant. Les figures réelles, c'est-à-dire les corps, *y* changent avec le temps et avec le lieu. Les courbures constantes des espaces méteuclidiens sont donc aussi éloignées de la réalité que l'est l'homogénéité de l'espace euclidien".

[24] The contracted passage given above is from Peter Pesic's English translation (Riemann, 2007/1853). The full passage in German reads as follows: "Die nach Grosse und Richtung bestimmte Ursache (beschleunigende Schwerkraft); welche nach 3. in jedem Punkte des Raumes stattfindet, suche ich in der Bewegungsform eines durch den ganzen unendlichen Raum stetig verbreiteten Stoffes, und zwar nehme ich an, dass die Richtung der Bewegung der Richtung der aus ihr zu erldarenden Kraft gleich, und ihre Geschwindigkeit der Grosse der Kraft proportional sei. Dieser Stoff kann also vorgestellt werden als ein physicher Raum; dessen Punkte sich in dem geometrischen bewegen".

This passage suggests that Riemann may have endorsed the recognition of two distinct notions of space; on one hand a real space as a kind of substance, a physical field, and on the other hand a geometrical, conceived space with respect to which we define and measure the motion or curvature of the physical space.

Although Kant himself had not considered the possibility of non-Euclidean spaces, it is worth noting that Kant does make a similar distinction between "empirical" and "absolute" space according to which the latter, which exists "for the sake of the possibility of experience", is considered as "in itself nothing, and no object at all" (Kant, 1970/1786, p.16). We may here draw an analogy: whereas the motion of an empirical, relative space, for Kant, "presupposes in turn an enlarged space" in which it is moved, we could say that the curvature or inhomogeneity of some empirical space presupposes a homogeneous space with respect to which this curvature would be defined. In this way we would arrive by abstraction at Delboeuf's notion of a homogeneous space, lacking curvature, not as a physical thing, but as a reflection of a fundamental feature of the scientific method.

Overall, it seems fair to characterise Delboeuf's philosophy of geometry as broadly neo-Kantian; moreover, unlike the well established neo-Kantianism of today (Bitbol et al., 2009; Friedman, 2001) which was to some extent inaugurated by Helmholtz, Delboeuf's approach offers a path forwards which avoids parting ways with Euclidean apriorism.

3. Reflections and discussions

3.1. Mobility or Leibniz shifts?

Readers may have noticed the contrast between Delboeuf's arguments for scale-invariance and Helmholtz's requirements for the possibility of congruence. Helmholts's notion of congruence is empirically grounded, it depends upon the *physically realisable* motions of natural solids. Dilations of natural solids, on the other hand, are not physically realisable. On what grounds, then, do we assert the possibility of *similarities*?

In the physically realised motion of a natural solid, the given body is known to have been moved because it has changed its relation to other bodies. *Has it moved in geometrical space?* That is entirely a matter of convention since this space is a creation of our minds. The only space in which we know it to be moved is the *relative* space defined, and perhaps *conditioned*, by the surrounding bodies. By verifying the empirical possibility of congruence, we have only shown that this *physical* space – the relative space conditioned by surrounding bodies – is approximately isogenous. We have shown nothing of geometrical space.

The translations that are analogous to the dilations imagined by Wallis, Laplace, Delboeuf, and others, are not translations of single bodies with respect to others, they are *Leibniz shifts*: motions of all bodies in the universe with respect to geometrical space itself. These motions lack any physical meaning, and take place only in our minds. We are not concerned with motions of some bodies with respect to others, since such motions could only tell us of the properties of bodies. We want to know about the properties of space itself, and, since we conceive this space as *passive*, we may assert that the relations among bodies should be invariant under Leibniz shifts. This condition tells us with certainty that the curvature of geometrical space is constant. Further, we also assert the invariance of the relations among bodies under *universal dilations*, which tells us that geometrical space must be Euclidean.

The impossibility of dilating natural solids with respect to one another informs us that the real, physical space they mutually inhabit possesses a definite scale, but tells us nothing about the properties of an ideal space. The converse is also true: the possibility of dilating all bodies with respect to an imagined space ensures that it must be Euclidean, but says nothing of the physical properties bodies possess

with respect to one another. One class of motions is proper to the one, the other to the other, but the two are not interchangeable. As we have seen above, in the *Metaphysical Foundations of Natural Science*, Kant explicitly distinguishes between absolute space, which is an ideal form, and empirical or relative space which are those spaces in which we perceive objects to be moved (Kant, 1970/1786, p. 16–17). Therefore Helmholtz's claim that the possibility of congruence of natural solids implies anything about Kant's forms of intuition, is simply mistaken.

If we admit that Helmholtz's empiricist method can only tell us of the geometry of a real or physical space, then history has vindicated Delboeuf's objection to this. As we saw in Section 2.3.3, Delboeuf insisted that real space has a curvature which is "different at each of its points and varies there at each instant", therefore the non-Euclidean spaces of constant curvature "have no special quality to represent real space better than the [Euclidean]"(Delboeuf, 1894a, p.372). On the other hand, If we wish to determine the properties of a space conceived as a pure form in the Kantian sense, then, once again, it is Delboeuf's method which is more appropriate.

3.2. Russell's relative angles objection

The tendency to assert the relativity of position to the neglect of the relativity of magnitude is epitomised by Russell (1897, 1898). Towards the end of the 19th Century, like Helmholtz and Poincaré, Russell defended the notion that space should be of constant curvature, but that we had no criterion by means of which to favour Euclidean geometry. In his "*Essay on the foundations of geometry*", Russell emphasises above all the *passivity of space*, arguing that "Geometry depends throughout on the irrelevance of causation" (Russell, 1897, p.112). On this basis he endorses Helmholtz's axiom of *free mobility*, yet rejects Delboeuf's argument for space's "homogeneity". Russell's rejection is founded on a peculiar understanding of the "space-constant" of a non-Euclidean space of constant curvature, according to which it is not itself a magnitude but a "standard of comparison" between magnitudes (Russell, 1897, §79, §98). Accordingly, a dilation of all magnitudes with respect to space would leave no observable change since the value of the space constant would be transformed along with all the magnitudes.

Since Delboeuf himself had unfortunately died abruptly in 1896 at the age of 64, prior to the publication of Russell's book, his view was defended instead by the French philosopher-mathematician Louis Couturat. In a response essay to Russell, Couturat remarks that Russell's conception of the space-constant is at odds with other statements Russell makes such as: "although measurement and the judgment of quantity express the result of comparison, yet the terms compared must exist before the comparison" (Russell, 1897, §164, ft.2). Since a given magnitude can be measured with respect to the space constant, this implies that the space constant must be a real thing (Couturat, 1898)[25]:

> It is therefore not correct to say that the spatial constant is not a quantity, how could it not be one, since all the quantities of the corresponding space can be related to it? We can only measure a quantity by a quantity of the same kind; now, if we can measure all the magnitudes of a space in relation to the constant of this space, this constant must obviously be itself a magnitude, indeed a spatial magnitude.

In a follow-up article that same year, Russell appears to have acknowledged Couturat's critique, however he raises a new argument against Delboeuf's ideas (Russell, 1898):

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[25] Translation of: "Il n'est donc pas exact de dire que la constante spatiale n'est pas une grandeur comment n'en serait-elle pas une, puisque toutes les grandeurs de l'espace correspondant peuvent lui être rapportées? On ne peut mesurer une grandeur que par une grandeur de même espèce; or, si l'on peut mesurer toutes les grandeurs d'un espace par rapport à la constante de cet espace, il faut évidemment que cette constante soit elle-même une grandeur, et une grandeur spatiale".

I come now to the principal argument in favour of the a priori character of Euclidean space, namely the argument which derives from the impossibility of an absolute magnitude. For this discussion, it will suit me better to adopt Delboeuf's terminology than to insist upon my own. [...] The question is: Can it be demonstrated that homogeneity is an a priori property of space?

On this point, a strong argument in my favour is derived, I believe, from the absolute magnitude of angles. Those who affirm it to be evident a priori that the sides of a triangle can be lengthened in a given ratio without altering the angles, ought to hold, it seems to me, that it is equally possible to alter all the angles in a given ratio without altering the sides. But that, we know, is impossible in all Geometries. If the logically relative nature of all magnitude is admitted, I cannot see why the argument would apply only to linear dimensions, and not to angles, which are equally magnitudes.

This rather strange argument by Russell might be dismissed as an idiosyncrasy were it not that others have claimed the same. It is independently repeated by Torretti (2012/1978, p.297) for instance, and even Poincaré (1898) raises the same objection in his own essay of 1898:

It is absurd, they say, to suppose a length can be equal to an abstract number. But why? Why is it absurd for a length and not absurd for an angle?

Since Couturat appears not to have addressed this objection, and Delboeuf did not live to see it, we will here show, on their behalf, that it is founded on a blatant misconception.[26]

The rebuttal to this argument is suggested by Russell's formulation of it; *the relativity of angles is impossible in all geometries.* Why is this?

One does not need to compare two angles in order to measure them. The angle of intersection of two lines is already a relation, a relation between the directions of these two lines. An angle denotes an objective relation between two directions, not a relation of an object to space. While we may assert the relativity of directions (based on the isotropy of space), we cannot assert the relativity of angles. Similarly, while we may assert the relativity of magnitudes (based upon the homogeneity of space), we cannot assert the relativity of ratios between magnitudes. We cannot assert the relativity of angles therefore, for the very same reason that we cannot assert the relativity of ratios between magnitudes.

To Russell, the fact that angles seem "tied absolutely to their magnitude" is evidence for the existence of an *angular space-constant*:

We have an angular space-constant in every space, namely the four right angles. [...] angles are tied absolutely to their magnitude, and cannot be conceived as all expanded in a given ratio. We cannot therefore infer, from the fact that magnitude is relative, the impossibility of a space-constant.

But Russell makes a mistake when he imagines that 2π is an angular space constant; Russell is implicitly assuming here that angles denote relations of things to space. In fact, they do not, rather, as we have seen, angles denote relations between the directions of lines. Moreover, angles can also be understood as ratios of magnitudes if we remember that a *radian* is simply the quotient of the length of an arc by a radius of a circle.[27] The claim that 2π (which is the angle between two lines that have the same direction) represents an angle constant everywhere in

space is as meaningless as the claim that at each of its points, space is endowed with a *ratio constant* of 1, i.e. the ratio between two lengths of the same magnitude which might form a standard with respect to which other ratios can be defined. But we naturally understand that a ratio is not a relation of an object to space, but rather, represents a relation between two magnitudes. Delboeuf's assertion of the homogeneity of space is, above all, an assertion of its *ideality*. Real things do not have real relations to space, but only to one another. If space possessed a scale constant, then bodies would be endowed with real relations to space, this property of space would become measurable, and space would no longer be *ideal*.

As relations, angles are measurable in and of themselves, whereas the magnitudes of bodies must be compared to one another if they are to be measured. This is the source of their relativity. Epistemically, all measurement is a relation between two given things.[28] By means of this principle, it is easy to rule out the possibility of using non-Euclidean geometries as forms for phenomena; for if it is claimed that we live, or should represent ourselves to live, in a space of constant positive or negative curvature, we must ask upon what reason this claim is based. There are two options:

1. If this choice is grounded upon some empirical observations, suppose for instance that we live on the surface of a hypersphere; then according to the principle stated above, this hypersphere – whose curvature is measurable with respect to real objects – must be a real object itself.
2. If on the other hand it is not grounded empirically, rather, this geometry is being used purely in its capacity as a form; then there is no reason to choose it over the Euclidean. In this case, Poincaré's simplicity criterion rules it out, moreover, a compensatory field would need to be invented to abrogate its needless effects.

In both cases we find therefore that the geometries of constant, nonzero curvature, cannot – despite what was asserted by Helmholtz and others – be used as forms for phenomena on account of the relativity of all measurement of sizes. This argument naturally carries over to the more general geometries of changing curvature as well, which are even less competent to be conceived as forms. We are forced to the conclusion that space, as pure form, must be Euclidean.[29]

3.3. Synthetic knowledge and the passivity of space

Given the weakness of the *relative angles* objection, it is surprising that Poincaré approves it in his essay of 1898 (Poincaré, 1898). A decade later, however, Poincaré's views concerning the relativity of magnitude seem to have changed. In book II of his volume *Science and Method* (Poincaré, 1914/1908), amid comments concerning the relativity of space, Poincaré affirms the relativity of magnitude, citing Delboeuf as the principal proponent of this idea[30]:

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[26] Note, Russell and Couturat would go on to have an extended multi-year correspondence (Schmid, 1983).

[27] Note also that this definition only hold in a Euclidean space, and that in a non-euclidean space the angle will need to be defined as a ratio of infinitesimals.

[28] Note that this principle is very similar to that which Couturat raised in order to refute Russell's claim that the space constant is not a real quantity (as discussed above).

[29] As complementary to this argument for the Euclidean nature of space based on the relativity of magnitude, it is interesting to note that, as Eisenthal (2024) has very recently pointed out, only Euclidean geometry is consistent with the relativity of velocity due to geodesic deviation. In the conclusion of his essay, Eisenthal writes: "If absolute motion is regarded as impossible in principle – if space is not the kind of thing that objects can move with respect to – then this idea can be leveraged as a metaphysical argument against the possibility of space having a constant curvature".

[30] This is a famous passage which has recently drawn attention due to its suggestion of the possibility of scale-invariant cosmological models. See for instance Gryb and Sloan (2021).

there is another [sense of the relativity of space], upon which Delboeuf [*sic*] has particularly insisted. Suppose that in the night all the dimensions of the universe become a thousand times greater: the world will have remained similar to itself, giving to the word similitude the same meaning as in Euclid, Book VI. Only what was a meter long will measure thenceforth a kilometer, what was a millimeter long will become a meter. [...] When I awake to-morrow morning, what sensation shall I feel in presence of such an astounding transformation ? Well, I shall perceive nothing at all. [...]

Poincaré uses this idea to deny that we can have knowledge of absolute magnitudes (Poincaré, 2015/1913, p.414), but does not go on to discuss Delboeuf's argument for the Euclidean nature of space.

Poincaré's deliberate avoidance of Delboeuf's thesis may be connected to his personal enthusiasm about non-Euclidean geometries as mathematical objects of study. In a well-known anecdote, he recounts how, out of the blue, upon stepping onto an omnibus in Coutances, it suddenly hit him with full clarity that "the transformations [he] had used to define the Fuchsian functions were identical with those of non-Euclidean geometry" (Poincaré, 2015/1913, p.417). This realisation would have taken place at some time before 1880 (Gray, 1997). It is plausible that Poincaré's personal involvement with the development of non-Euclidean geometries drove him away from the defenders of Euclidean apriorism, who, at the time, were largely considered to be a reactionary force, opposed to those that were creatively driving the progress of knowledge. This may have led him to seek out conventionalism as a mid-way compromise between empiricism and apriorism.

Elsewhere in his writing, however, Poincaré has based his "*principle of relative motion*" on an affirmation of the "passivity of space" (Poincaré, 2015/1913, p.83). Russell too, who, even more than Poincaré, defended non-Euclidean geometries of constant curvature, affirmed that space is *passive*. But what is the root of this intuition of space's passivity?

We have seen above that if we base some notion of the passivity of space on the invariance of bodies when they are moved with respect to one another, we will only have learned of the (approximate) passivity of a *physical* space (see Section 3.1). This physical space does not permit the dilations of individual bodies, therefore it may be non-Euclidean. But *who are we to say that this physical space is passive?* Why should physical space allow for the possibility and mobility of rigid bodies? Even if we put aside the complications raised by modifications due to heat, surely the question of whether a natural solid retains the same relations among its parts when it is moved with respect to other physical bodies is one that should be answered by empirical science. Indeed this is what was done, through the recognition of the *equivalence principle* – which, in the division between force and inertia, places gravity on the side of inertia and (chrono-)metricity – Einstein fulfilled Riemann's ideas and showed once and for all that the physical space which governs those motions which have traditionally been called "inertial" is not passive, but *dynamical*.

But what then of our intuition of the passivity of space, and the corollary *relativity of motion*? This law, and the intuition underlying it, can only be based in a truly Kantian conception of space, a space abstracted from all contingent phenomena. Notions of space's passivity, the relativity of magnitudes, and the relativity of motion are pervasive in Poincaré's works. Poincaré at times justifies these ideas on the basis that contrary hypotheses would be "repugnant to the mind" (Poincaré, 2015/1913, p. 107–109), but he does not discuss why we feel this repugnance. I propose that these intuitions we have of space, of the relativity of motion and of magnitudes, are rooted in this same methodological concept of space's homogeneity which is responsible for providing foundations for Euclidean geometry.

Quite apart from space and its geometry, we saw in Section 1.1 that for Poincaré, the inductive method, which allows a formula to generalise over an infinity of cases, is the "veritable type of the synthetic a priori judgment" since it is "inaccessible to analytic demonstration and to experience" (Poincaré, 2015/1913, p.39). But what is it that makes this generalisation possible? Is it the mathematical concept of infinity? An infinite set which is not *ordered*, which is not in some respect homogeneous, does not permit of generalisations. It is not the notion of infinity that allows for reasoning by recurrence, it is the concept of symmetry, of the absolute mathematical invariance of some property under transformations. It is through the "process of abstraction", described by Delboeuf, that we gain access to this form of reasoning; and it is from this basic notion of *sameness*, which we find in the notion of homogeneity and of symmetry in general, and which, as Plato emphasises in the *Phaedo* (Hackforth et al., 1972), is not known to us empirically, that Poincaré's "synthetic a priori" propositions arise.

4. Morals for scientific methodology

As we have seen, unlike other neo-Kantian influences on contemporary philosophy of space and time, that of Cassirer and those of Helmholtz and Poincaré, Delboeuf's neo-Kantianism affirms the apriority of Euclidean geometry. It may be argued that Delboeuf's ideas, though perhaps interesting, have little relevance to present-day physics and philosophy of physics, since, after all, Delboeuf did not work directly in physics, his ideas had little or no influence on the development of Einstein's theories, and, unlike Cassirer, his philosophy was not developed in response to these paradigm-shifting ideas. In what respect, then, should we take this account seriously?

We have seen throughout this essay that the central insight upon which Delboeuf grounds his philosophy of geometry – that is, of the relativity of magnitude – was not unique to him. It dates back at least to Wallis in 1663, and was recognised by a variety of significant physicists and philosophers over the centuries. We even saw that Leibniz embarked upon a project very similar to Delboeuf's in his attempt to find stable foundations for geometry. Moreover, we have seen that Delboeuf's account is defensible in the context of the philosophies of geometry that were present at the time, it stands up to Russell and Poincaré's fallacious *relative angles* objection (Section 3.2), it is both more Kantian and more self-consistent than Helmholtz's allegedly neo-Kantian approach (Section 3.1), and it even resolves certain problems in the foundations of Poincaré's philosophy of mathematics (Section 3.3).

Concerning the applications of these ideas to physics, it is clear that Delboeuf can only contribute on the *methodological* side of things. We may propose a strict distinction between *geometrical* space, conceived as a form, and *physical* space (or space–time), conceived as part of the content of this form. Indeed, as Lehmkuhl (2014) has shown, even Einstein showed some restraint against unequivocally accepting the geometrical interpretation of his theory; for instance in his review of Meyerson's *La déduction relativiste*, he writes (Einstein & Metz, 1928; Lehmkuhl, 2014):

> The fact that the metric tensor is denoted as "geometrical" is simply connected to the fact that this formal structure first appeared in the area of study denoted as "geometry". However, this is by no means a justification for denoting as "geometry" every area of study in which this formal structure plays a role, not even if for the sake of illustration one makes use of notions which one knows from geometry.

Since the advent of general relativity, a vast literature of flat space alternatives or subtle modifications has been proposed.[31] This literature raises a wide array of methodological advantages of working in

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[31] See for instance: Arminjon (2002), Broekaert (2005), Cavalleri and Spinelli (1980), Davies and Falkowski (1982), Deser (1970), Dicke (1957), Fang and Fronsdal (1979), Gupta (1954), Huggins (1962), Kraichnan (1955), Lasenby, Doran, and Gull (1998), Logunov and Mestvirishvili (1985), Mittelstaedt and Barbour (1967), Nachtmann, Schmidle, and Sexl (1969), Ogievetsky and Polubarinov (1965), Pitts and Schieve (2001), Rosen (1940a, 1940b), Thirring (1961), Weinberg (1964a, 1964b).

flat space, including: (1) the recovery of a well-defined local gravitational energy and of global energy conservation laws (Logunov & Mestvirishvili, 1985; Rosen, 1940a, 1940b),[32] (2) greater consistency with methods in particle physics (Lasenby et al., 1998), (3) avenues towards unification with particle physics and prospects of developing a theory of quantum gravity (Dicke, 1957; Lasenby et al., 1998; Pitts & Schieve, 2001), (4) the possibility of implementing various interpretations of Mach's principle (Dicke, 1957; Sciama, 1953), (5) the development of simplified models of gravity which reproduce some of the basic results of Einstein's theory (Arminjon, 2002; Broekaert, 2005). Many of these models explicitly appeal to Poincaré's notion of the conventionality of geometry to justify their methods, however, given the findings of the present paper, we suggest that Delboeuf's forgotten arguments may also help to provide a philosophical grounding for these flat space approaches.

The topics that Delboeuf's writings raise, however, are most relevant to certain recent developments in the physics and philosophy of cosmology. In recent years, Julian Barbour has been attempting to extend the Machian research program to encompass a requirement for the scale-invariance of cosmological models (Barbour, 2010). If we refuse to accept the existence of epistemically inaccessible absolutes, then the universe must consist only in the relative configuration of its parts—i.e., its *shape*. This way of thinking has led to the development of the theory of *Shape Dynamics* (Barbour, 2012; Mercati, 2018). If we recognise that the shape of a body consists of the internal relations amongst its parts, while its size is an external relation to other bodies, then the universe as a whole, which has no external reference possesses only a shape. The central insight discussed in this essay – which was recognised by Wallis and Delboeuf – is that the reciprocal independence of shape and size implies the Euclidean nature of space. This essay may help provide grounds for Barbour et al.'s use of Euclidean space as a background for their models.

It is only in the context of cosmological models, rather than in the study of subsystems of the cosmos, that transformations of all bodies with respect to space, i.e. *Leibniz shifts* or transformations by *similarity*, can be considered. Outside of shape dynamics, the requirement for the invariance of dynamics under similarity transformations in cosmology has been called "*dynamical similarity*", and it is a growing area of research in cosmology (Bravetti, Jackman, & Sloan, 2022; Gryb & Sloan, 2021; Sloan, 2018). We hope that the ideas discussed in the present paper will help to provide some philosophical context and justification for these cosmological speculations.

Acknowledgements

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[32] In such theories, the principle of conservation of energy would appear, not as a contingent empirical fact, but as a guiding methodological principle.

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