John Wallis,
Demonstration of Euclid's Fifth Postulate

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by J. Fay, using claude ai

It is known that some of the Ancients, and some of the Moderns, have charged Euclid with the fault of having postulated his Fifth Postulate — or, as others call it, the Eleventh Axiom, or (as Clavius numbers it) the Thirteenth Axiom — to be granted without demonstration: a thing (so they judge) which he ought to have demonstrated. This especially because he assumes, as though clear by its own light, something concerning *straight* lines which is not true of lines in general. For although what he affirms holds universally of straight lines — namely,

> *If a straight line, falling on two straight lines, makes the interior angles on the same side less than two right angles, those two lines, continued to infinity, will meet on the side where the angles are less than two right angles —*

it does not, nevertheless, hold universally of curved lines. For two curves, or a straight and a curved line, can perpetually approach one another and yet never meet.

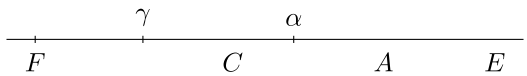
But those who fault Euclid in this matter themselves, for the most part (at any rate those I have so far examined), assume in his place other propositions — which seem to me no easier to be granted than the very thing Euclid postulates. And they not infrequently dash upon the very rock they wish to avoid: presuming, namely, as undoubtedly true of straight lines what is not true of lines in general — as we have shown elsewhere.

For my own part, I would not find it difficult to grant Euclid what he postulates. Not only because the demonstrations adduced by others stumble on the same fault they blame in him — or at any rate are no clearer than what they themselves postulate — but because either this very thing seems to require postulating, or else something in its place; or finally because, even if it were most fully granted to be demonstrable, it is customary to hold as Principles not only things which can in no way be demonstrated, but also things so clear by their own light that they require no demonstration. For it is certain that some of the remaining Axioms can also be demonstrated; which, if needed, is not hard to show.

Yet, since I see so many before me have attempted its demonstration — as though they thought it stood in need of some demonstration — it has seemed good to add our own contribution as well; to test whether the demonstration we propose may be less liable to objections than those hitherto offered.

We thus approach the proposed demonstration by way of certain Lemmas which need to be set out beforehand.

I. If a finite straight line, lying on an infinite straight line, is continued in its direction, the continuation will also lie on the same infinite line.



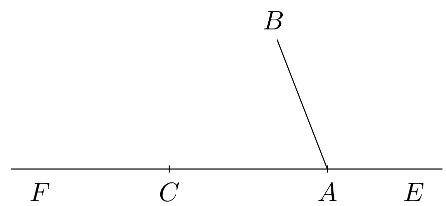
Let $EACF$ be an infinite straight line, and on it the finite straight line AC lying; and let it be continued in its direction to γ. I say that the whole $AC\gamma$ — that is, AC continued — lies on the same infinite line ACF. For since, by hypothesis, ACF is one straight line, CF is placed in a straight direction with AC. But also (because AC is continued in a straight direction to γ) $C\gamma$ is placed in a straight direction with AC. And so it lies on CF itself. (For everyone presumes that, at the same extremity of the same straight line, distinct straight lines cannot be placed in a straight direction; this is almost the same as what Clavius, from Proclus, makes his Tenth Axiom.) But also AC lies on the same ACF, by hypothesis. Therefore the whole $AC\gamma$ — that is, AC continued — lies on the same infinite line ACF. Which was to be demonstrated.

II. If a finite straight line, lying on an infinite line, is understood to be advanced in its direction by any amount whatever, the advanced segment will also lie on the same infinite line.

Let AC be a finite straight line, lying on the infinite line AF. And let it be understood to be advanced in its direction on the side of C: namely, the point A being advanced to α, and C to γ. I say that the line $\alpha\gamma$ — that is, AC advanced — lies on the same infinite line AF. For $C\gamma$ lies on AC continued. (Since C is supposed to be advanced in a straight direction; that is, on AC continued.) Therefore on ACF the infinite line (by Lemma 1). Similarly α also will lie on the same AC (at any rate continued). (For it is supposed that the point A is advanced in a straight direction along this same AC line.) Therefore on the same infinite line ACF. And the same will similarly be shown of any other intermediate point of the advanced line AC. Therefore the whole $\alpha\gamma$ — that is, the line AC advanced — lies on the same infinite line ACF. Which was to be demonstrated. And the same may likewise be shown if the same line AC were advanced on the side of A.

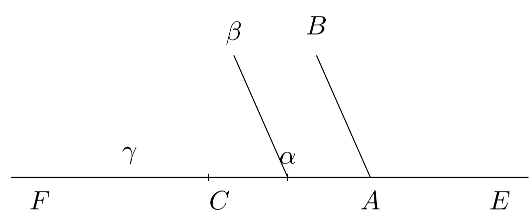
Nor is it any obstacle here that the motion of a straight line does not yet appear to have been used by Euclid in demonstrations, nor mentioned among his Postulates. For by the same reasoning by which he afterwards uses the Motion of a Circle in the definition of a Sphere, and the motion of a Triangle in the definition of a Cone, and the motion of a Rectangle in the definition of a Cylinder, he could equally have used the motion of a Straight line, if there had been need, in demonstrations. And this Archimedes, Apollonius, and other Geometers do everywhere. Indeed Euclid himself (right at the beginning) in the demonstration of the fourth proposition uses, through *ἐφαρμογή* [superposition], both the motion of straight lines and the angle held invariant — to the extent, namely, required for superposition. (Nor in any other sense is a Straight line understood to be moved in our proposition.) Add that in the Third Postulate (*Given a centre* *and an interval, to describe a circle*), this very thing is presupposed. For it is supposed (in the construction of the Circle) that the plane of the circle is described by the sweep of a Radius (one of its extremities remaining at the Centre). I mention this lest I should appear (neglecting the strictness of Euclidean demonstrations) to be introducing new Postulates here, beyond those which Euclid admits.

III. If on a finite line lying on an infinite line a straight line stands, making an Angle with it, this same line makes the same Angle with the infinite line.



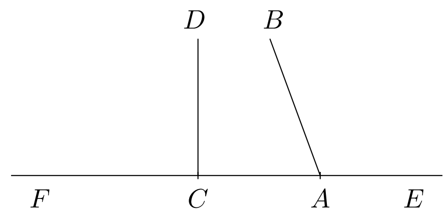
Let EAF be an infinite straight line, and on it the finite line AC lying; and let the line AB stand on it, making an angle with it, BAC: I say that the same line AB makes the same Angle with the infinite line AF. For since the line AC lies on the line AF, and BA is common; BAC and BAF are (by congruence) the same angle. Which was to be demonstrated.

IV. If on an infinite line a finite line, lying on it, is advanced in its direction; and a straight line standing on it is carried along simultaneously with the angle unvaried: this latter line makes everywhere, with that infinite line, the same (or equal) angles.



On the infinite line EAF let the finite line AC lying on it be advanced in its direction; and let the line AB standing on it be carried along simultaneously, the angle BAC being invariant; until, AC having been advanced into the position $\alpha\gamma$, AB is simultaneously carried into $\alpha\beta$: I say that the angle $\beta\alpha F$ is equal to the angle BAC (or BAF). For since AC advanced is now in $\alpha\gamma$ — which (by Lemma 2) lies on AF the infinite line — and (by hypothesis) the angle BAC, that is $\beta\alpha\gamma$, remains invariant; and since (by Lemma 3) this invariant angle congrues first with the angle BAF, and then with the angle $\beta\alpha F$: therefore the angles BAF, $\beta\alpha F$ will be equal to one another. Which was to be demonstrated. The same will likewise be shown of the angle $\beta\alpha A$, equal to the angle BAE.

V. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles: the exterior angle (adjacent to either) is greater than the opposite interior angle.

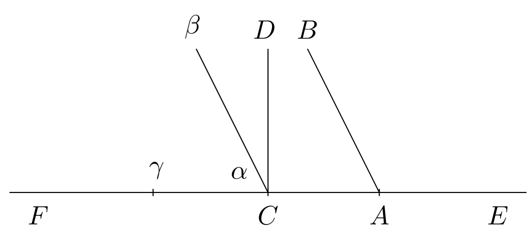


On the two straight lines AB, CD, let the line ACF falling make the interior angles, on the same side, BAC DCA, less than two right angles: I say that the exterior angle — say DCF (adjacent to the interior DCA) — is greater than the opposite interior BAC. For together the two angles DCA, DCF are equal to two right angles (by I.13). But together the two interior angles DCA, BAC are (by hypothesis) less than two right angles. The common angle DCA being therefore subtracted from each side, the remainder DCF will be greater than the remainder BAC. Which was to be demonstrated.

I assume here (in the demonstration) Proposition 13 of Book I of the *Elements*; for although it is posterior to the Fifth Postulate, it is nevertheless prior to Proposition 29 of Book 1, in whose demonstration this postulate is first employed; and so the true place (or at any rate a sufficiently apt one) for demonstrating it is after Proposition 28. All the preceding propositions could

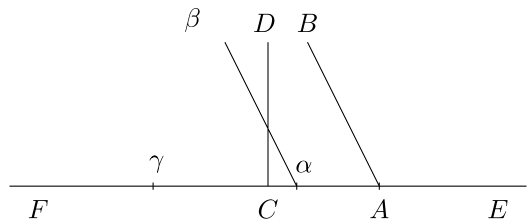
be duly used in the demonstration, or any of them; and Proposition 13 of Book 1 could itself be more quickly demonstrated (as another Lemma serving the present proposition) if there were need.

VI. With the same suppositions: if the line interjacent, AC, is advanced in its direction into the position $\alpha\gamma$ (so that the point α now coincides with C); and at the same time AB is carried along (the angle BAC remaining invariant) into the position $\alpha\beta$: I say that the whole line $\alpha\beta$ — that is, AB advanced — falls outside CD.



For since (by Lemma 2) $\alpha\gamma$, that is $C\gamma$, lies on CF; and (by Lemmas 3 & 4) the angle BAC, that is BAF, is equal to the angle $\beta\alpha F$, that is βCF; and since (by Lemma 5) the angle BAC is less than the angle DCF: also βCF will be less than the same angle DCF. And so the line $C\beta$ — that is, the line $\alpha\beta$ — falls outside CD entirely. (Entirely, I say; for it cannot meet CD anywhere else than at the point C, by the last postulate or axiom, lest two straight lines should enclose a space.) Which was to be demonstrated.

VII. With the same suppositions: I say that the line $\alpha\beta$, that is AB advanced, cuts the line CD before the point α reaches C.



For since (by Lemma 6) when the point α reaches C the whole line $\alpha\beta$ has crossed the line CD: it must be that it has crossed either all at once, or part by part. But it cannot have crossed all at once; for then the line $\alpha\beta$ would at some moment lie on CD itself, and so the angle DCF would congrue with the angle $\beta\alpha F$ (a greater with a lesser) — which is impossible. Therefore it crosses part by part; that is, it cuts the line CD at some point; namely when some part of it has crossed but not the whole. This is (by Lemma 6) before the point α has reached the point C. Which was to be demonstrated.

VIII. I presume finally (the nature of Ratios being taken as known, and the definition of Similar Figures), as a Common Notion:

That to any given Figure, a Similar one of any magnitude is possible.

For this (on account of continuous quantities being infinitely divisible, and equally infinitely augmentable) seems to flow from the very nature of Quantity: namely, that any figure can continually — with the species of the figure preserved — be both diminished and augmented to infinity.

And in truth (although unobserved, and perhaps not themselves noticing) everyone presumes this; and with others, Euclid himself. For when he postulates, *Given a centre and an interval, to describe a circle*, he presumes a circle of any magnitude, or with any radius, to be possible: and what he presumes can be done, he postulates that you can do. And although it would not be equally fair to postulate, that, given any figure, you can (not yet having been taught how) construct a similar one on a given line: nevertheless that it is possible for this to be done is no less to be presumed of any figure than of the Circle. For it is not on account of anything peculiar to the Circle, above other figures, that, with its species preserved, it can be continually augmented or diminished into any magnitude; but on account of the nature of the Continuum, which is common to the remaining figures along with the Circle. And so the same can equally be presumed of them — that, with their species preserved, continual augmentation or diminution to infinity is possible.

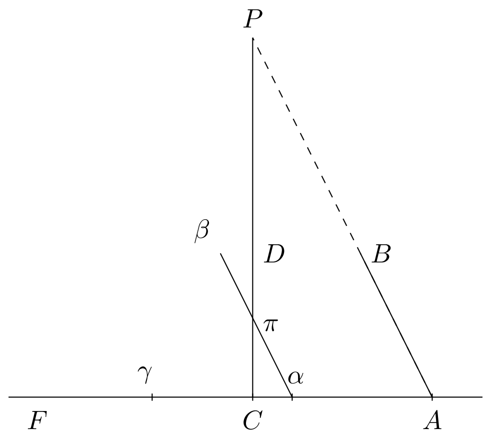
Nor does it stand in the way of our Presumption that the definition of Proportionals, and (which presupposes it) the definition of Similar Figures, had not yet been delivered by Euclid (but the one in Book V, the other in Book VI, to be delivered later): for Euclid could, if it had seemed expedient, have placed both at the beginning of Book I.

IX. By means of these Lemmas I demonstrate the principal proposition as follows: namely,

If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles: those two straight lines, produced to infinity, will meet on the side where those two angles are less than two right angles.

Let those two straight lines be AB, CD, on which let the infinite line ACF fall, making interior angles on the same side BAC, DCA less than two right angles taken together:

I say that those two straight lines AB, CD, if they be produced to infinity, will meet — and indeed on those parts of the line AF where those two angles lie. For let the interjacent line AC be understood to be advanced in its direction along the infinite line ACF; and let AB, standing on it, be carried along simultaneously, the angle BAC being kept invariant, until $\alpha\beta$ — that is, AB advanced — cuts the line CD (according to Lemma 7), say at the point π.



There will then be a triangle $\pi C\alpha$; to which (by Lemma 8) a similar triangle of any magnitude is possible. It is therefore possible, upon the line CA, to construct a Triangle similar to the triangle $\pi C\alpha$ set up on the base $C\alpha$. Let this then be understood to be done: and let that triangle be PCA.

Nor is it any obstacle here that, *upon a given line, to construct a triangle similar to a given one*, Euclid had not yet taught. For many things are everywhere presumed in the Σχολίοις [Scholia] to the demonstrations of Theorems (although in the construction of Problems it is otherwise), and supposed as done, the manner of whose geometrical accomplishment has not yet been delivered. Such are: *between two given lines, two mean proportionals*; *a straight line equal to the perimeter of a Circle*; and innumerable others. The demonstrations of Theorems proceed no differently in respect of these than if the geometrical construction were fully established. And if anyone were to set out to prove the Antarctic Circle equal to the Arctic by *Congruence* — since, if Centre be understood applied to centre, and Plane to plane, Perimeter would congrue to perimeter (on account of the equality of the radii) and circle to circle: no one would reject this as an illegitimate demonstration on the ground that it has not yet been geometrically established how anyone could move the Antarctic circle up to the Arctic. It suffices that these circles are so constituted that, if they were applied, they must necessarily congrue. The demonstration proceeds here in just the same way — provided it be established that what is understood as done, namely the triangle PCA, can be made — as we now pursue it.

Since therefore PCA is a triangle: the two lines CP, AP meet one another (by the definition of a Triangle) at the point P. And since the triangle PCA is similar to the triangle $\pi C\alpha$ (by construction): the angles too, taken respectively one by one, are equal to the angles (by the definition of similar rectilinear figures). Therefore the angle PCA is equal to the angle $\pi C\alpha$, that is, to the angle DCA. And so the line CP lies on the line CD itself (produced). (For if the line CP lay beyond or short of it, the angle PCA would be greater or less than the angle DCA, which is shown to be equal.) Likewise, the angle PAC is equal to the angle $\pi\alpha C$. But to that same angle $\pi\alpha C$, that is, to the angle $\beta\alpha F$, the angle BAF, or BAC, is equal (by Lemmas 3 & 4); therefore the angle BAC is also equal to the angle PAC. Therefore the line AP lies on the line AB produced: (for if it lay beyond or short of it, the angles BAC, PAC would be unequal, which are demonstrated to be equal). Therefore the same line is AP and AB produced. Likewise the same line is CP and CD produced. But AP and CP meet at the point P (as has just been shown); therefore AB and CD produced also meet. And indeed at the same point P; that is, on those parts of the line EAF where those two angles are less than two right angles. Which was to be demonstrated.

I have pursued this demonstration in accordance with the strictest laws of demonstrating, in imitation of Euclid; so that there should be nothing which even a Severe censor could complain of as wanting to a just demonstration. But I am so far from blaming Euclid for not having demonstrated this himself, that I would not have blamed him if he had postulated still more things without demonstration. Suppose, for instance, he had postulated (with Archimedes) *that a Straight Line is the shortest of all between the same points*: (in which case there would have been no need, after nineteen propositions, to demonstrate that *two sides of a Triangle, taken together, are greater than the remaining one*). And other things which are clear by their own light.

But Euclid seems to have set himself this aim: that from the fewest possible postulates he should demonstrate the rest by the firmest of consequences. (Whence it has come about that he not infrequently sets himself the trouble of proving things which everyone would have conceded gratis.) And indeed in every proof (in whatever subject matter) something must be presumed. For unless from things presumed (whether previously conceded, or previously proved) no proof is made. These presumptions, although by other Writers (treating of other subjects) they are not customarily set out explicitly (as is done by Euclid) — such things they nevertheless tacitly presume,

though unobserved. And Euclid himself, too, in the course of the work, presumes everywhere — beyond these things expressly recorded (as the chief and more notable ones) — other things, whether manifest from inspection of the Diagram or otherwise; but such as no one would deny. Such are (what is everywhere presumed) that the Whole is equal to all the Parts taken together (whence what is proved equal to all the Parts taken together is concluded to be equal to the Whole). Likewise, that what is demonstrated true in individual Cases is universally true (namely, that what holds of the Right-angled, the Acute-angled, and the Obtuse-angled Triangle, holds of every Rectilineal Triangle; on the ground that there are no other Rectilineal Triangles). Likewise, that 1 and 1 are 2; that 4 and 1 are five; and similar things, which the curious Reader will observe everywhere, but no one will blame. (I pass over that he presumes the motion of a Plane in the definition of the Sphere, Cone, and Cylinder — which he had neither defined nor postulated.) And indeed if he had still further — either tacitly presumed or expressly postulated — more such things (which are clear by their own light), he would not on that account be blameworthy; far less when he has only postulated that *Two Straight Lines (in the same plane) being Convergent will eventually meet.*

But I add this also. Euclid (as it seems to me at least) explicitly (ῥητῶς) interposes this Postulate not so much for the sake of what *is* postulated as for the sake of what is *not* postulated. (Lest he should appear to have postulated more than is just.) For who is there who, if this were tacitly presumed (without even drawing attention to it), would not concede it? But if perhaps anyone should notice that Euclid, in what follows, while demonstrating, tacitly presumes certain stated lines, if they are produced, will eventually meet; and should doubt whether, and how far, this is to be permitted: Euclid here explicitly warns him *what* it is, and to *what extent*, that he postulates may be conceded to him. Not, indeed, that he postulates this concerning *any* kind of Lines, but at any rate of *straight* ones. (For it is certain that Curves can be so continued for any length whatever as never to meet.) Nor concerning *all* Straight lines, but those *lying in the same plane*. (For straight lines not in the same plane, even ones somewhat approaching one another, can certainly be so situated that, if continued, they pass each other mutually, are afterwards separated, nor meet in the interim.) Nor concerning *any number whatever* of such, but at any rate concerning *two*. (For, by way of example, Three straight lines, even if placed in the same plane, will not at once concur all at the same point: but any two of them in the same point; the third perhaps not in the same point as those two together, but with one of them at a second point, with the other at a third — not all in the same.) Nor again concerning straight lines *however* placed in the same plane, but concerning *convergent* ones; that is, those upon which, if a straight line falls, it makes the interior angles on the same side together less than two right angles. (For those which are not convergent, but parallel, will not meet.) Nor again that these two already meet, but that, if they be produced, they will meet. Nor again *however* produced (say, if while continued they become curved), but if they be produced *in a straight direction*. Nor again *however little* they be produced, but if produced *to infinity* (at any rate as far as need be). Nor again, if they be produced on *one side only* (namely, the side on which they more diverge); but if they are so produced *on both sides*, they meet. Nor yet that they meet *on both sides*; but they meet (at any rate on *one* side). Not indeed indifferently on either, but on the side where those two angles are less than two right angles. And this Statement, so limited, there is no one who should not concede. Nor is Euclid only to be exculpated; he is to be praised, for having so distinctly set out what he means. Namely:

> *Lines (not any whatever, but at any rate) straight (not any number, but) two (not however situated, but) in the same plane (and not so however, but such that) upon which, a straight line falling, makes the interior angles on the same side less than two right angles, (do not yet, perhaps, meet, but) if they be produced (not however, but) in a straight direction, (and not however little, but) to infinity: will eventually meet (not, however, on both sides of the falling line, nor on either indifferently, but) on the side where those two angles are less than two right angles. Which I judge to have been done with the best counsel.*

And let these things suffice for the vindication of Euclid.

Finis.